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THE OPEN-ENDED COAXIAL LINE: A RIGOROUS VARIATIONAL TREATMENT

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SUMMARY

The open-ended coaxial line with a ground plane or effectively infinite flange has attracted much interest in recent years as a device for non-destructive measurement of complex permittivity, eg in bio-medical applications. Existing theoretical treatments of this configuration all involve approximations; this paper presents a rigorous treatment using a variational approach.

THE OPEN-ENDED COAXIAL LINE: A RIGOROUS VARIATIONAL TREATMENT

T E Hodgetts

INTRODUCTION

This paper presents a rigorous theoretical analysis of the open-ended coaxial line with an infinite ground plane or flange ~~(Fig 1)~~ radiating into a semi-infinite lossy medium. This configuration has become popular for measuring the complex permittivity of liquids and bio-medical specimens, because it does not require a sample of a special shape. However, the theoretical analysis has previously only been performed by approximate methods; the best of these seems to be the point-matching technique, employed by Mosig et al [1], (discussed with attention to numerical detail by ~~Symm [2]~~), but there are several others in the literature (see the bibliography of [3]). The present treatment employs the variational methods described by Collin [4] and applied to the behaviour of enclosed coaxial-line discontinuities by Whinnery et al [5], Somlo [6], Bianco et al [7] and the present author [8].

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THE ELECTROMAGNETIC FIELD EQUATIONS

We begin by outlining the forms of the fields relevant to the problem. These have been discussed at length by many writers (e.g Jones [9], Kerns and Beatty [10] and the present author [8]), so this section will be little more than a statement of notation without further formal references.

A region of space which is linear, isotropic and homogeneous, and contains no free charges or current-carrying elements, will support a monochromatic electromagnetic field specified by the six space components of the conventional complex field vectors $\underline{E}$ and $\underline{H}$, which are related by the complex monochromatic forms of Maxwell's equations:

$$\left.\begin{array}{ll} \mathrm{curl}\ \underline{E} + j\omega\mu\underline{H} = \underline{0} & \mathrm{div}\ \underline{H} = 0 \\ \mathrm{curl}\ \underline{H} - j\omega\epsilon\underline{E} = \underline{0} & \mathrm{div}\ \underline{E} = 0 \end{array}\right\} \qquad (1)$$

In eqns (1), ω is the angular frequency, so that the time variation of the fields is as $\exp(j\omega t)$ where j is the square root of (−1), and μ and ϵ are the permeability and complex permittivity of the medium in the space region. The complex permittivity is defined in the usual way, having a real part equal to the ordinary absolute permittivity and an imaginary part equal to $(-\sigma/\omega)$ where σ is the conductivity of the medium. In our problem there are two regions, bounded by an interface between them (at the end-plane of the coaxial line) and by various perfect conductors (the flange and the cylinders of the coaxial line); eqns (1) apply separately to each region.

Both regions have rotational symmetry about the axis of the coaxial line, so it is convenient to use cylindrical polar co-ordinates (ρ, φ, z). When this is done, it turns out that the six space components of $\underline{E}$ and $\underline{H}$ de-couple into two sets, (H_φ, E_ρ, E_z) and (E_φ,H_ρ,H_z). The field in the coaxial line far from its end necessarily has components belonging only to the first of these sets (at normal working frequencies); and it is possible to satisfy all the conditions at the interface and the bounding conductors without introducing components in the second set, which can therefore be discarded.

Equations (1) comprise eight scalar equations. Remembering that we have rotational symmetry - so that all the field components are independent of φ - and discarding the four equations which relate to the irrelevant (null) components (E_φ, H_ρ, H_z), we have

four significant equations, one of which is an immediate consequence of the other three. (This redundancy follows directly from eqns (1); if we take the divergences of the two (vector) equations involving curls, the other two (scalar) equations follow immediately.) The three remaining equations are

$$\left.\begin{aligned} E_\rho &= \left[\frac{j}{\omega\epsilon}\right]\left[\frac{\partial H_\varphi}{\partial z}\right] \\ E_z &= \left[-\frac{j}{\omega\epsilon}\right]\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi)\right] \end{aligned}\right\} \qquad (2)$$

$$\frac{\partial^2}{\partial\rho^2}(\rho H_\varphi) - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi) + \frac{\partial^2}{\partial z^2}(\rho H_\varphi) + k^2(\rho H_\varphi) = 0 \qquad (3)$$

where $k^2 = \omega^2\mu\epsilon$ (as usual) and the unit vectors ($\hat{\underline{\rho}}$, $\hat{\underline{\varphi}}$, $\hat{\underline{z}}$) form a right-handed set (see Fig 1).

MODES IN THE COAXIAL LINE

The tangential electric field on any perfectly-conducting surface is zero. The directions tangential to the cylinders of the coaxial line are $\underline{\varphi}$ and $\underline{z}$; hence, from eqns (2),

$$\frac{\partial}{\partial\rho}(\rho H_\varphi) = 0 \quad \text{at } \rho = r \text{ and } \rho = R \qquad (4)$$

where r and R are the radii of the inner and outer conductors.

This condition has no explicit z-dependence, so eqn (3) can be solved subject to it by separating the variables ρ and z. The equation in z is harmonic or exponential; the equation in ρ is a close relative of Bessel's equation. A separation constant appears, which is constrained to take only particular values (by eqn (4)). The complete field for $z < 0$ (within the line; see Fig 1) can now be written down (after [8] and [9]) as follows.

The equation

$$J_0(k_i r)Y_0(k_i R) - J_0(k_i R)Y_0(k_i r) = 0 \qquad (5)$$

has a set of real, positive and simple zeros k_i, where i is an integer numbering the zeros in ascending magnitude, and J_0 and Y_0 are Bessel's functions of the first and second kinds of order 0 (Watson [11]). For each i we define a constant C_i satisfying

$$C_i = J_0(k_i r)/Y_0(k_i r) = J_0(k_i R)/Y_0(k_i R) \qquad (6)$$

(which is consistent with eqn (5)), and we also define three functions of ρ, thus

$$Z_\alpha(k_i\rho) = J_\alpha(k_i\rho) + C_i Y_\alpha(k_i\rho) \tag{7}$$

where α is one of the integers 0, 1 or 2 (2 being only rarely required); the Z_α will only be used with arguments of the form $(k_i\rho)$ so that the required value of C_i is always obvious. From eqns (6) and (7), the mode constants k_i now satisfy

$$Z_0(k_i r) = Z_0(k_i R) = 0 \tag{8}$$

for all i. Also, from [8] or [9],

$$\frac{d}{dx}(xJ_1(x)) = xJ_0(x) \text{ and } \frac{d}{dx}(xY_1(x)) = xY_0(x),$$

$$\text{so} \quad \frac{d}{d(k_i\rho)}\left[(k_i\rho)Z_1(k_i\rho)\right] = (k_i\rho)Z_0(k_i\rho) \tag{9}$$

The field components then are

$$H_\varphi(z<0) = \frac{1}{\eta_A\rho}a_0\left[e^{-jk_A z} - \Gamma_A e^{jk_A z}\right] + \sum_{i=1}^{\infty}\left[-\frac{j\omega\epsilon_A}{\gamma_i}\right]a_i e^{\gamma_i z} Z_1(k_i\rho) \tag{10a}$$

$$E_\rho(z<0) = \frac{1}{\rho}a_0\left[e^{-jk_A z} + \Gamma_A e^{jk_A z}\right] + \sum_{i=1}^{\infty} a_i e^{\gamma_i z} Z_1(k_i\rho) \tag{10b}$$

$$E_z(z<0) = \sum_{i=1}^{\infty}\left[-\frac{k_i}{\gamma_i}\right]a_i e^{\gamma_i z} Z_0(k_i\rho) \tag{10c}$$

where the suffix A denotes region A in Fig 1, and

$$k_A^2 = \omega^2\mu_A\epsilon_A \quad \text{and} \quad \gamma_i^2 = k_i^2 - k_A^2 \quad \text{and} \quad \eta_A = \sqrt{\frac{\mu_A}{\epsilon_A}} \tag{11}$$

with k_A, η_A and the γ_i being positive (if ϵ_A is purely real), or having positive real parts (if ϵ_A is complex). (It is assumed that the frequency is low enough for all the γ_i^2 to have positive real parts, so that only the ordinary coaxial transmission-line mode propagates freely, all other modes decaying exponentially away from the plane z = 0 even in the lossless case.) The remaining parameters, Γ_A, a_0 and the a_i, are to be determined later. This representation may be compared with that in [2].

THE FIELD IN THE UNBOUNDED DIELECTRIC REGION

The specification of the unbounded field is considerably more complicated. Symm [2] takes it directly from a result given by Lewin [12]; however, as we have started from first principles (eqns (1)), it is appropriate to take the treatment somewhat further back so as to show more clearly the connection between the results for the bounded and unbounded cases, and to justify the transformation of variables used by Symm [2].

We begin with a theorem quoted in [12] from Baker and Copson [13]; the proof of this is an exercise in pure mathematics, not physics, and the reader is referred to [13] for the details. (To apply the theorem we change temporarily to Cartesian co-ordinates, retaining the z-direction and taking the x-z plane as the $\varphi = 0$ plane; see Fig 1).

Let ν be a solution of Helmholtz' equation in region B of Fig 1; that is, $\nabla^2\nu + k_B^2\nu = 0$, where $k_B^2 = \omega^2\mu_B\epsilon_B$ and k_B is positive (for real ϵ_B) or has a positive real part and a negative imaginary one (for complex ϵ_B) (compare eqns (11) and the text immediately following them). Let ν further have no singularities in the positive-z half-space, and satisfy the radiation condition at infinity; and let the function $(\partial\nu/\partial z)_{(z=0)}$ (defined, if necessary, as the limit as z tends to zero from above) be denoted by $\psi(x,y)$. Then, in the positive-z half-space,

$$\nu(x', y', z') = -\frac{1}{2\pi} \int\int \psi(x,y) \frac{\exp(-jk_B D)}{D} dxdy \qquad (12)$$

where D is the positive square root of $((x - x')^2 + (y - y')^2 + z'^2)$ and the double integral extends over the whole z = 0 plane. (As before, a harmonic time factor $\exp(j\omega t)$ has been suppressed.)

Now, region B satisfies the same symmetry conditions as region A, from which it was deduced that the field in region A has only components (H_φ, E_ρ, E_z). We infer that the same is true of the field in region B, since the two regions are connected by continuity conditions on the tangential components of $\underline{E}$ and $\underline{H}$ at the z = 0 plane. We cannot, however, apply eqn (12) immediately to the reduced set of field components, because the critical components H_φ and E_ρ do not satisfy Helmholtz' equation; but the Cartesian components of $\underline{E}$ and $\underline{H}$ do satisfy it. It is easiest to establish this result and deduce its consequences if we follow the treatment in [9], where it is shown that, subject to the assumptions already made, the electromagnetic field in a region such as B can be expressed in terms of two scalar functions satisfying the damped wave equation (normally taken as the magnitudes of Hertzian vectors with directions fixed along the z-axis). Using the implied time-harmonic factor with the results in [9], the Cartesian field components are

$$E_x = \frac{\partial^2\Pi}{\partial x\partial z} - j\omega\frac{\partial M}{\partial y} \qquad \mu_B H_x = \frac{\partial^2 M}{\partial x\partial z} + j\omega\mu_B\epsilon_B\frac{\partial\Pi}{\partial y}$$

$$E_y = \frac{\partial^2\Pi}{\partial y\partial z} + j\omega\frac{\partial M}{\partial x} \qquad \mu_B H_y = \frac{\partial^2 M}{\partial y\partial z} - j\omega\mu_B\epsilon_B\frac{\partial\Pi}{\partial x}$$

$$E_z = \frac{\partial^2\Pi}{\partial z^2} + k_B^2\,\Pi \qquad \mu_B H_z = \frac{\partial^2 M}{\partial z^2} + k_B^2\,M \tag{13}$$

where the complex scalars Π and M now satisfy Helmholtz' equation.

We now wish to prove that

$$\frac{\partial H_y}{\partial z} = -\,j\omega\epsilon_B E_x \quad \text{and} \quad \frac{\partial H_x}{\partial z} = j\omega\epsilon_B E_y \tag{14}$$

Given that H_z is zero in region B, these results in fact follow directly from eqns (13). A less direct, but more physically instructive, way of establishing eqns (14) is by transforming into the (ρ, φ, z) co-ordinates and discarding derivatives of Π and M with respect to φ – the $\underline{E}$ and $\underline{H}$ fields are independent of φ, so common sense would be contradicted if Π and M were not also independent of it. We then find that the components (H_φ, E_ρ, E_z) depend only on Π, and (E_φ, H_ρ, H_z) depend only on M; as this second set of components is null, we may discard M, and eqns (13) then simplify to forms from which eqns (14) follow without further assumptions.

It is also clear from eqns (13), whether or not the terms in M are discarded, that H_x and H_y satisfy Helmholtz' equation – because Π and M do so, and therefore so do all their Cartesian derivatives (as can be shown by differentiating Helmholtz' equation and inverting the order). Hence we may set H_x or H_y for ν in eqn (12), taking the corresponding $(\partial\nu/\partial z)$ from eqns (14); then

$$\left.\begin{aligned} H_y(x',y',z') &= +\frac{j\omega\epsilon_B}{2\pi}\iint E_x(x,y,0)\,\frac{\exp(-jk_B D)}{D}\,dxdy \\ H_x(x',y',z') &= -\frac{j\omega\epsilon_B}{2\pi}\iint E_y(x,y,0)\,\frac{\exp(-jk_B D)}{D}\,dxdy \end{aligned}\right\} \tag{15}$$

Now, by taking components in the various orthogonal directions in Fig 1 it is easy to show that

$$H_\varphi = H_y\cos\varphi' - H_x\sin\varphi' \quad \text{(in the primed co-ordinates)}$$

and

$$\left.\begin{aligned} E_x &= E_\rho\cos\varphi - E_\varphi\sin\varphi \\ E_y &= E_\rho\sin\varphi + E_\varphi\cos\varphi \end{aligned}\right\} \text{(in the unprimed co-ordinates)}$$

using $x = \rho \cos \varphi$, $y = \rho \sin \varphi$ and similar primed equations. We substitute from these equations for E_x and E_y in eqns (15); then we multiply the first of eqns (15) by ($\cos \varphi'$) and the second by ($- \sin \varphi'$), add, and combine the two integrals into one. We thus obtain, remembering that E_φ is zero,

$$H_\varphi(x',y',z') = \frac{j\omega\epsilon_B}{2\pi} \int\int E_\rho(x,y,0) \cos(\varphi - \varphi') \frac{\exp(-jk_B D)}{D} dxdy$$

We now complete the conversion back to cylindrical polar co-ordinates. Since E_ρ is tangential to the $z = 0$ plane over which we are integrating, it vanishes for all radii ρ less than r or greater than R, since these parts of $z = 0$ are occupied by perfect conductors (Fig 1); and the differential area ($dxdy$) transforms into ($\rho d\varphi d\rho$), as is well known. The last equation then becomes

$$H_\varphi(\rho',\varphi',z') = \frac{j\omega\epsilon_B}{2\pi} \int_r^R \int_0^{2\pi} E_\rho(\rho,\varphi,0)\cos(\varphi - \varphi') \frac{\exp(-jk_B D)}{D} \rho d\varphi d\rho \quad (16)$$

where now $D^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + z'^2$, and the dependence of E_ρ on φ and of H_φ on φ' is only formal. This is the result given by Symm [2].

CURRENT, VOLTAGE, ADMITTANCE AND REFLECTION COEFFICIENT

It is possible to carry on the analysis from here in terms of the principal mode's "voltage reflection coefficient" Γ_A (introducted in eqns (10)) or in terms of current, voltage and admittance. The latter representation is quite logical (since we have a terminated two-conductor system) and is more satisfying to microwave engineers; but it requires a lengthy discussion to justify it rigorously (eg. [10]). For convenience we shall proceed in terms of the reflection coefficient; the link with admittance can then be made using the standard equation ([8],[10])

$$\frac{Y}{Y_o} = \frac{1 - \Gamma_A}{1 + \Gamma_A} \quad (17)$$

where Y_0 is the characteristic admittance of the coaxial line and is $2\pi/(\eta_A \ell n(R/r))$.

INTEGRAL RELATIONS AND ORTHOGONALITY

It is well known that the modes in modal expansions like eqns. (10) satisfy integral relations which make them mutually orthogonal. The necessary relations for our problem are quoted in [8] from [11] and are

$$\int \left[\frac{1}{\rho} \right]^2 \rho d\rho = \ell n\rho$$

$$\int Z_1^2(\theta\rho)\rho d\rho = \frac{1}{2} \rho^2 \left[Z_1^2(\theta\rho) - Z_0(\theta\rho)Z_2(\theta\rho)\right]$$

$$\int \left[\frac{1}{\rho} \right] Z_1(\theta\rho)\rho d\rho = -\frac{1}{\theta} Z_0(\theta\rho)$$

$$\int Z_1(\theta_1\rho)Z_1(\theta_2\rho)\rho d\rho = \frac{\left[\theta_2\rho Z_1(\theta_1\rho)Z_0(\theta_2\rho) - \theta_1\rho Z_1(\theta_2\rho)Z_0(\theta_1\rho)\right]}{(\theta_1^2 - \theta_2^2)} \quad (\theta_1 \neq \theta_2) \tag{18}$$

The last two of these are the orthogonality relations properly so called, as can be seen by taking the integration from r to R, setting the θ's equal to one or two of the k_i and using eqn (8).

VARIATIONAL FIELD REPRESENTATIONS

We are now ready to tackle the heart of the problem. Let the field components H_φ and E_ρ at z = 0 be denoted by the forms $\mathcal{H}_\varphi$ and $\mathcal{E}_\rho$. By allowing z to tend to 0 in eqns (10) and (16) we obtain two representations of $\mathcal{H}_\varphi$; these must be equal, since the tangential components of $\underline{H}$ are continuous across the plane z = 0, but in some of what follows it will be convenient to distinguish them, as $\mathcal{H}_\varphi^+$ and $\mathcal{H}_\varphi^-$ according to the sign of z. $\mathcal{H}_\varphi$ and $\mathcal{E}_\rho$ are functions only of ρ, because of the cylindrical symmetry.

From eqns (10) and (16), with some interchanges of ρ and ρ' for convenience,

$$\begin{aligned} a_0(1 - \Gamma_A) &= \eta_A\rho\left[\mathcal{H}_\varphi^- + \sum_{i=1}^{\infty}\left[\frac{j\omega\epsilon_A}{\gamma_i}\right]a_i Z_1(k_i\rho)\right] \\ &= \eta_A\rho\left[\mathcal{H}_\varphi^+ + \sum_{i=1}^{\infty}\left[\frac{j\omega\epsilon_A}{\gamma_i}\right]a_i Z_1(k_i\rho)\right] \\ &= \eta_A\rho\left[\sum_{i=1}^{\infty}\left[\frac{j\omega\epsilon_A}{\gamma_i}\right]a_i Z_1(k_i\rho)\right. \\ &\quad \left. + \frac{j\omega\epsilon_B}{2\pi}\int_r^R\int_0^{2\pi} \mathcal{E}_\rho(\rho')\cos(\varphi - \varphi')\,\frac{\exp(-jk_B D_0)}{D_0}\,\rho' d\varphi' d\rho'\right] \end{aligned} \tag{19}$$

where $D_0^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi - \varphi')$.

It is possible for the integrand in eqn (19) to become singular; to deal with this problem a suitable approach to the limit will be used.

Now, from eqns (10), (18) and (8), with interchanges of integration and summation,

$$a_0(1 + \Gamma_A)\ell n(R/r) = \int_r^R \left[E_\rho - \sum_{i=1}^{\infty} a_i Z_1(k_i\rho)\right] d\rho$$

$$= \int_r^R E_\rho \, d\rho - \sum_{i=1}^{\infty} a_i \int_r^R Z_1(k_i\rho)d\rho$$

$$= \int_r^R E_\rho d\rho \text{ ; and similarly}$$

$$a_i \int_r^R Z_1^2(k_i\rho)\rho d\rho = \int_r^R Z_1(k_i\rho)\rho\left[E_\rho - \frac{a_0(1+\Gamma_A)}{\rho} - \sum_{\substack{i'=1 \\ i'\neq i}}^{\infty} a_{i'} Z_1(k_{i'}\rho)\right]d\rho$$

$$= \int_r^R E_\rho Z_1(k_i\rho)\rho d\rho \quad .$$

Substituting these results for a_0 and a_i into eqn (19) gives us

$$\left[\frac{2\pi}{\eta_A \ell n(R/r)}\right]\left[\frac{1 - \Gamma_A}{1 + \Gamma_A}\right]\left[\int_r^R E_\rho d\rho\right] =$$

$$2\pi\rho\left[\sum_{i=1}^{\infty} \left[\frac{j\omega\epsilon_A}{\gamma_i}\right] Z_1(k_i\rho)\left[\int_r^R Z_1^2(k_i\rho)\rho d\rho\right]^{-1}\left[\int_r^R E_\rho Z_1(k_i\rho)\rho d\rho\right]\right.$$

$$\left. + \left[\frac{j\omega\epsilon_B}{2\pi}\right] \int_r^R \int_0^{2\pi} E_\rho(\rho')\cos(\varphi - \varphi')\left[\frac{\exp(-jk_B D_0)}{D_0}\right]\rho' d\varphi' d\rho'\right]$$

This equation contains the quantity $(2\pi/(\eta_A \ell n(R/r)))((1 - \Gamma_A)/(1 + \Gamma_A))$, which by eqn (17) we are entitled to call the terminating admittance Y of the coaxial line. Let us now multiply the equation by an arbitrary function $\overline{E}_\rho$ of ρ and integrate from r to R; we then have

$$Y\left[\int_r^R E_\rho d\rho\right]\left[\int_r^R \overline{E}_\rho d\rho\right] =$$

$$2\pi \sum_{i=1}^{\infty} \left[\frac{j\omega\epsilon_A}{\gamma_i}\right]\left[\int_r^R Z_1^2(k_i\rho)\rho d\rho\right]^{-1}\left[\int_r^R E_\rho Z_1(k_i\rho)\rho d\rho\right]\left[\int_r^R \overline{E}_\rho Z_1(k_i\rho)\rho d\rho\right]$$

$$+ \ j\omega\epsilon_B \int_r^R \int_r^R \int_0^{2\pi} \bar{E}_\rho(\rho) E_\rho(\rho')\cos(\varphi - \varphi') \left[\frac{\exp(-jk_B D_0)}{D_0}\right] \rho\rho' d\varphi' d\rho' d\rho \qquad (20)$$

In eqn (20) the function E_ρ is the exact form of the radial electric field E_ρ at the plane z = 0. We desire now to choose the function $\bar{E}_\rho$ so that we obtain the best possible value for Y when we use (as we inevitably must) an approximation to E_ρ. To do this, we consider the effect on eqn (20) of a small change in $\bar{E}_\rho$ or E_ρ. In the terminology of the calculus of variations (eg [4], [8]) the first variation of eqn (20) with respect to $\bar{E}_\rho$ is

$$Y\left[\int_r^R E_\rho d\rho\right]\left[\int_r^R \delta\bar{E}_\rho d\rho\right] + \delta Y\left[\int_r^R E_\rho d\rho\right]\left[\int_r^R \bar{E}_\rho d\rho\right] =$$

$$2\pi \sum_{i=1}^{\infty} \left[\frac{j\omega\epsilon_A}{\gamma_i}\right]\left[\int_r^R Z_1^2(k_i\rho)\rho d\rho\right]^{-1}\left[\int_r^R E_\rho Z_1(k_i\rho)\rho d\rho\right]\left[\int_r^R \delta\bar{E}_\rho Z_1(k_i\rho)\rho d\rho\right]$$

$$+ \ j\omega\epsilon_B \int_r^R \int_r^R \int_0^{2\pi} \delta\bar{E}_\rho(\rho) E_\rho(\rho')\cos(\varphi - \varphi') \left[\frac{\exp(-jk_B D_0)}{D_0}\right] \rho\rho' d\varphi' d\rho' d\rho$$

or $$\delta Y\left[\int_r^R E_\rho d\rho\right]\left[\int_r^R \bar{E}_\rho d\rho\right] =$$

$$\int_r^R d\rho\,\delta\bar{E}_\rho \left\{2\pi\rho \sum_{i=1}^{\infty} \left[\frac{j\omega\epsilon_A}{\gamma_i}\right] Z_1(k_i\rho)\left[\int_r^R Z_1^2(k_i\rho)\rho d\rho\right]^{-1}\left[\int_r^R E_\rho Z_1(k_i\rho)\rho d\rho\right]\right.$$

$$+ \ j\omega\epsilon_B\rho \int_r^R \int_0^{2\pi} E_\rho(\rho')\cos(\varphi - \varphi')\left[\frac{\exp(-jk_B D_0)}{D_0}\right]\rho' d\varphi' d\rho'$$

$$\left. - \ Y\left[\int_r^R E_\rho d\rho\right]\right\} \qquad (21)$$

The integrand in brace brackets here vanishes, on account of the equation preceding eqn (20); hence δY is zero for variations of $\bar{E}_\rho$. From the symmetry of eqn (20), we can immediately deduce a counterpart to eqn (21) with E_ρ interchanged with $\bar{E}_\rho$ (and ρ interchanged with ρ'), and it then follows that setting $\bar{E}_\rho = E_\rho$ makes δY vanish for variations of E_ρ also. This is the optimum form, since first-order errors in E_ρ then only cause second-order errors in Y. The vanishing variation property can also be used in reverse, after Ritz ([4],[8]), as follows. Putting $\bar{E}_\rho = E_\rho$ in eqn (20) gives us

$$Y\left[\int_r^R E_\rho d\rho\right]^2 =$$

$$2\pi \sum_{i=1}^{\infty} \left[\frac{j\omega\epsilon_A}{\gamma_i}\right] \left[\int_r^R Z_1^2(k_i\rho)\rho d\rho \right]^{-1} \left[\int_r^R E_\rho Z_1(k_i\rho)\rho d\rho \right]^2 \tag{22}$$

$$+ j\omega\epsilon_B \int_r^R \int_r^R \int_0^{2\pi} E_\rho(\rho)E_\rho(\rho')\cos(\varphi - \varphi') \left[\frac{\exp(-jk_B D_0)}{D_0}\right] \rho\rho' d\varphi' d\rho' d\rho$$

If an approximate form for E_ρ, containing adjustable parameters, is inserted into eqn (22), it will represent E_ρ as well as it can when the first variations of the equation with respect to the adjustable parameters all vanish. Before applying this technique, however, we will first transform the triple integral in the equation into a more tractable form.

REPRESENTATION OF THE PROPAGATION FACTOR

The factor $(\exp(-jk_B D_0)/D_0)$ - sometimes called the propagation factor - usually appears in analyses of this kind, and has a large body of literature associated with it in classical electromagnetics and optical diffraction theory, including the Sommerfeld dipole problem which is perhaps the present problem's nearest relative (see Banos [14] for a very full discussion and bibliography). The integral transform we need is, in fact, discussed by Bateman [15] in the context of Sommerfeld's work. However, it is unsafe simply to quote the result from [15], as a branch-point integral is involved and considerable care is needed. Accordingly, we shall prove it directly, following the method given by Jeffreys and Jeffreys [16].

We note first that $(\exp(-jk_B D_0)/D_0)$ is the limit as z' tends to zero of $(\exp(-jk_B D)/D)$ (see eqns (12) and (16)), and it is this latter form which we shall represent, so that the limit can be taken later (see the comment following eqn (19)).

Now,

$$\exp(-\ell z')J_0\left[D_0\surd(k_B^2 + \ell^2)\right] = \frac{1}{2\pi} \int_0^{2\pi} \exp\left[-\ell(z' + jD_0\cos\alpha) + jk_B D_0 \sin\alpha\right] d\alpha \tag{23}$$

where D_0, z' and k_B have the meanings already used, ℓ is a general complex number with a non-negative real part, and J_0 is the Bessel's function of the first kind and order 0 already introduced. (The sign of the square root on the left-hand side is of no consequence, since J_0 is an even function.) This result is easily proved by removing the factor $\exp(-\ell z')$ from both sides, expanding the remaining exponential integrand as an absolutely convergent series with infinite convergence radius, and integrating it term by term; the equation then reduces to

$$J_0\left[D_0\surd(k_B^2 + \ell^2)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2^n n!)^2} \left[D_0^2(\ell^2 + k_B^2)\right]^n$$

which is true by definition (eg. [16]).

Next, we consider the integral

$$\int_0^\infty \frac{\exp(-z'(s^2 - k_B^2)^{\frac{1}{2}})}{(s^2 - k_B^2)^{\frac{1}{2}}} J_0(sD_0)sds = I \text{ (say)} \tag{24}$$

in which the path of integration is the positive real axis, and we introduce the temporary requirements that z' is strictly positive and k_B is strictly complex; the square root has its natural definition (approximately equal to s for large s on the integration path). If we change the variable of integration to this natural square root, which we will call ℓ, we then have

$$I = \int_{jk_B}^{\infty} \frac{\exp(-z'\ell)}{\ell} J_0\left[D_0\surd(k_B^2 + \ell^2)\right].\ell d\ell$$

where the lower limit of integration is fixed by continuity (since $(s^2 - k_B^2)$ has a positive imaginary part for all real s) and the square root in the argument of J_0 causes no trouble as remarked above. Since also the ℓ in the denominator is cancelled by the ℓ in the numerator, we have an integrand which is well-behaved in the entire complex finite ℓ-plane; this makes it unnecessary to specify the path of integration in detail, and (more importantly) allows us to substitute from eqn (23) and interchange the order of integration of the resulting double integral, which also has a well-behaved integrand; so

$$I = \frac{1}{2\pi} \int_0^{2\pi} \int_{jk_B}^{\infty} \exp\left[-\ell(z' + jD_0\cos\alpha) + jk_BD_0\sin\alpha\right] d\ell d\alpha$$

The integration over ℓ can be performed at sight, remembering that z' is strictly positive so that $\exp(-z'\ell)$ tends to zero as ℓ tends to infinity; we then have

$$I = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(-jk_Bz' + k_BD_0\cos\alpha + jk_BD_0\sin\alpha)}{z' + jD_0\cos\alpha} d\alpha$$

This integral is transformed by the substitution $\beta = \exp(j\alpha)$, which (contrary to appearances) does not introduce a branch point (because all the occurrences of α are trigonometric, except the differential $d\alpha$ which can be written as $d\beta/(j\beta)$). We then have

$$I = \frac{1}{2\pi j} \int_{(0)}^{(2\pi)} \frac{\exp(-jk_Bz' + k_BD_0\beta)}{\beta z' + \frac{1}{2}jD_0(\beta^2 + 1)} d\beta$$

where the path of integration is anti-clockwise around the unit circle in the complex β-plane. This integrand has two simple poles at the points $(jz'/D_0) \pm j\surd(1 + (z'/D_0)^2)$, and since z' is strictly positive it is clear that the contour only encloses the pole at $\beta = (jz'/D_0) - j\surd(1 + (z'/D_0)^2)$. We then obtain, from eqn (24),

$$\int_0^\infty \frac{\exp(-z'(s^2 - k_B^2)^{\frac{1}{2}})}{(s^2 - k_B^2)^{\frac{1}{2}}} J_0(sD_0)s\,ds = I = \frac{\exp(-jk_B D)}{D} \tag{25}$$

remembering that $D^2 = D_0^2 + z'^2$ from the definitions of D and D_0. This form may now be used in the triple integral in eqn (22), with the limit as z' tends to zero being taken later - this is physically justified by the continuity of the transverse field components at the plane z = 0, and is logically justified by the flow of the argument from eqn (16) to eqn (22). The triple integral accordingly becomes the limit as z' tends to zero of

$$\int_r^R \int_r^R E_\rho(\rho)E_\rho(\rho')\rho\rho' \left[\int_0^{2\pi} \cos(\varphi - \varphi') \int_0^\infty \frac{\exp(-z'(s^2 - k_B^2)^{\frac{1}{2}})}{(s^2 - k_B^2)^{\frac{1}{2}}} J_0(sD_0)s\,ds\, d\varphi' \right] d\rho' d\rho$$

INTEGRATION OVER φ'

We now consider the term in square brackets in the last expression, still maintaining the temporary requirements that z' is strictly positive and k_B is strictly complex. As before, we can remove the branch-point by temporarily transforming from the variable s to the previously defined variable ℓ. We may then invert the order of integration and transform back to the variable s, and the bracketed part of the expression becomes

$$[\ \ ] = \int_0^\infty \frac{\exp\left[-z'(s^2 - k_B^2)^{\frac{1}{2}}\right]}{(s^2 - k_B^2)^{\frac{1}{2}}} s \left\{ \int_0^{2\pi} J_0(sD_0)\cos(\varphi - \varphi')d\varphi' \right\} ds \tag{26}$$

The integral in brace brackets can be evaluated in closed form; and, remarkably, the integration variables ρ and ρ' are separated in the process. We remark first that D_0 depends on φ' only through $\cos(\varphi - \varphi')$, and this property is preserved for the whole integrand; and, moreover, if F represents an arbitrary well-behaved function we have

$$\int_0^{2\pi} F\left[\cos(\varphi - \varphi')\right]d\varphi' = \int_0^{2\pi} F\left[\cos(\varphi' - \varphi)\right]d\varphi'$$

$$= \int_{-\varphi}^{(2\pi-\varphi)} F(\cos\varphi')\, d\varphi'$$

$$= \int_0^{(2\pi-\varphi)} F(\cos\varphi')\, d\varphi' + \int_{-\varphi}^{0} F\left[\cos(2\pi + \varphi')\right]d(2\pi + \varphi')$$

$$= \int_0^{(2\pi-\varphi)} F(\cos\varphi')d\varphi' + \int_{(2\pi-\varphi)}^{2\pi} F(\cos\varphi')d\varphi'$$

$$= \int_0^{2\pi} F(\cos\varphi')d\varphi'$$

Hence the brace-bracketed integral may be written as

$$\int_0^{2\pi} J_0\left[\sqrt{(s\rho)^2 + (s\rho')^2 - 2(s\rho)(s\rho')\cos\varphi'}\right]\cos\varphi' d\varphi'$$

using the definition of D_0. Now J_0 has a triangular expansion given in [16] and [11] as (in our notation)

$$J_0\left[\sqrt{(s\rho)^2 + (s\rho')^2 - 2(s\rho)(s\rho')\cos\varphi'}\right]$$

$$= J_0(s\rho)J_0(s\rho') + \sum_{m=1}^{\infty} 2\cos(m\varphi')J_m(s\rho)J_m(s\rho')$$

where J_m denotes the Bessel function of the first kind and of order m. This series has an exponential tail, since the asymptotic formula in [9] for the Bessel functions of the first kind of fixed argument and large variable order shows that $J_m(w)$ (regarded as a function of m) decays exponentially to zero as soon as m significantly exceeds ($\frac{1}{2}ew$), e being the base of natural logarithms. We may therefore interchange the order of integration and summation after substituting this series into the expression above for the brace-bracketed integral in eqn (26). All the integrals over φ' then vanish, except the one where the index m is 1; and the brace-bracketed integral is simply given by

$$\{\ \} = 2\pi J_1(s\rho)J_1(s\rho')$$

Hence, from eqn (26) and the expression before it, the triple integral in eqn (22) is the limit as z' tends to zero of

$$\int_r^R \int_r^R E_\rho(\rho)E_\rho(\rho')\rho\rho' \int_0^\infty \frac{\exp(-z'(s^2 - k_B^2)^{\frac{1}{2}})}{(s^2 - k_B^2)^{\frac{1}{2}}} 2\pi s J_1(s\rho)J_1(s\rho')ds d\rho' d\rho$$

(This method of integrating over φ' is described by Morse and Feshbach [17] in the context of a similar problem.)

THE RITZ PROCEDURE

We now apply to the exact equation (22) the Ritz procedure of successive approximations described immediately after that equation. We want an approximate form for E_ρ which depends in a convenient way on adjustable parameters, approaches the exact form if enough parameters are used, and (for stability) satisfies the boundary and consistency conditions (implied by eqns (2) and (4)) at every approximation stage. The natural form is the one obtained by truncating the series expression for E_ρ at $z = 0$ from eqn (10), namely

$$\hat{E}_\rho(\rho) = \frac{1}{\rho}\hat{\alpha}_0 + \sum_{m=1}^{N} k_m \hat{\alpha}_m Z_1(k_m\rho) \qquad (27)$$

for the N^{th} approximation, where $\hat{\alpha}_0$ and the $\hat{\alpha}_m$ are the adjustable parameters (the factor k_m has been included for later convenience). The mode functions $1/\rho$ and the $Z_1(k_m\rho)$ are an orthogonal complete set (see [4] and [8]), so the form (27) can represent as accurately as we please in the mean any function in the relevant range of ρ and satisfying the relevant boundary conditions, if N is sufficiently large. At each approximation stage we substitute $\hat{E}_\rho$ into eqn (22), adjust every parameter for a vanishing first variation, and calculate the corresponding value of Y (say, Y_N for the N^{th} order). Since $\hat{E}_\rho$ depends linearly on its adjustable parameters, there is at each stage only one stationary point where the variations with respect to every parameter vanish; hence, since the exact E_ρ is also stationary in this sense, as N tends to infinity $\hat{E}_\rho$ tends to E_ρ in the mean and Y_N tends to Y, and so Y can be found by calculating suitable Y_N and extrapolating to infinity using a suitable procedure for accelerating convergence.

We substitute from eqn (27) first into the expression above for the triple integral in eqn (22). The result is a finite double sum of triple integrals, and the limit as z' tends to zero may be taken subsequently in each of these (provided they all remain convergent, which we shall see they do); we then have for this part of eqn (22)

$$\int_r^R \int_r^R \left[\frac{1}{\rho}\hat{\alpha}_0 + \sum_{m=1}^{N} k_m\hat{\alpha}_m Z_1(k_m\rho)\right]\left[\frac{1}{\rho'}\hat{\alpha}_0 + \sum_{n=1}^{N} k_n\hat{\alpha}_n Z_1(k_n\rho')\right] .$$

$$.\ \rho\rho' \int_0^\infty \frac{\exp(-z'(s^2 - k_B^2)^{\frac{1}{2}})}{(s^2 - k_B^2)^{\frac{1}{2}}}\, 2\pi s J_1(s\rho)J_1(s\rho')\,ds\,d\rho'\,d\rho$$

$$= \hat{\alpha}_0^2 I_{00} + \sum_{m=1}^{N} \hat{\alpha}_m^2 I_{mm} + 2\hat{\alpha}_0 \sum_{m=1}^{N} \hat{\alpha}_m I_{0m} + \sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} \hat{\alpha}_m\hat{\alpha}_n I_{mn}$$

where the integrals I satisfy $I_{mn} = I_{nm}$ and are given by

$$I_{00} = \int_r^R \int_r^R \int_0^\infty \frac{\exp(-z'(s^2 - k_B^2)^{\frac{1}{2}})}{(s^2 - k_B^2)^{\frac{1}{2}}} 2\pi s J_1(s\rho) J_1(s\rho') ds d\rho' d\rho \qquad (28a)$$

$$I_{0m} = \int_r^R \int_r^R \int_0^\infty \frac{\exp(-z'(s^2 - k_B^2)^{\frac{1}{2}})}{(s^2 - k_B^2)^{\frac{1}{2}}} (k_m\rho) Z_1(k_m\rho) . 2\pi s J_1(s\rho) J_1(s\rho') ds d\rho' d\rho \qquad (28b)$$

$$\underset{(m=n,\, m\neq n)}{I_{mn}} = \int_r^R \int_r^R \int_0^\infty \frac{\exp(-z'(s^2 - k_B^2)^{\frac{1}{2}})}{(s^2 - k_B^2)^{\frac{1}{2}}} (k_m\rho) Z_1(k_m\rho)(k_n\rho') Z_1(k_n\rho') . 2\pi s J_1(s\rho) J_1(s\rho') ds d\rho' d\rho \qquad (28c)$$

Still maintaining the temporary requirements that z' is strictly positive and k_B is strictly complex, we can once more remove the branch-point from each of the I-integrals by temporarily transforming from the variable s to the previously-defined variable ℓ; we may then invert the order of integration, transform back to the variable s, and integrate first over ρ' and ρ. The ρ' and ρ integrals are all special cases of eqns (18), with a Z-function which is a degenerate form with C_i (in the notation of eqn (7)) always zero. As a consequence of the fixed C_i it does not matter that the arguments of J_1 are not of the form $(k_i\rho)$ or $(k_i\rho')$, and on applying eqns (18) and recalling eqn (8) we obtain

$$\int_r^R J_1(s\rho) d\rho = s^{-1}\left[J_0(sr) - J_0(sR)\right]$$

$$\int_r^R (k_m\rho) Z_1(k_m\rho) J_1(s\rho) d\rho = \frac{(k_m r) Z_1(k_m r) s J_0(sr) - (k_m R) Z_1(k_m R) s J_0(sR)}{s^2 - k_m^2} \qquad (29)$$

with an identical pair of equations in ρ' and k_n. Using eqns (29), each of the I-integrals can be reduced to a one-dimensional integral over s.

To complete the application of eqn (27) to eqn (22), we substitute into the remaining integrals involving E_ρ in that equation; using the orthogonality relations in eqns (18), we obtain for the N^{th} order approximate admittance

$$Y_N \hat{\alpha}_0^2 \ell n^2(R/r) = 2\pi j\omega\epsilon_A \sum_{i=1}^{N} \hat{\alpha}_i^2 \left[\frac{k_i^2}{2\gamma_i}\right]\left[R^2 Z_1^2(k_i R) - r^2 Z_1^2(k_i r)\right]$$

$$+ \; j\omega\epsilon_B \left[\hat{\alpha}_0^2 \, I_{00} + 2\hat{\alpha}_0 \sum_{m=1}^{N} \hat{\alpha}_m \, I_{0m} + \sum_{m=1}^{N} \sum_{n=1}^{N} \hat{\alpha}_m \hat{\alpha}_n I_{mn} \right] \tag{30}$$

SIMPLIFICATION AND APPLICATION

Eqn (30) is homogeneous in the $\hat{\alpha}$'s; it also contains many appearances of factors $(k_i r)Z_1(k_i r)$ and $(k_i R)Z_1(k_i R)$. From eqns (7), (6) and (18) we have

$$(k_i r)Z_1(k_i r) \; = \; \left[Y_0(k_i r)\right]^{-1}(k_i r)\left[J_1(k_i r)Y_0(k_i r) - J_0(k_i r)Y_1(k_i r)\right]$$

$$= \left[Y_0(k_i r)\right]^{-1}(k_i r)\left\{J_0(k_i r)\frac{d\left[Y_0(k_i r)\right]}{d(k_i r)} - Y_0(k_i r)\frac{d\left[J_0(k_i r)\right]}{d(k_i r)}\right\}$$

The expression in braces is the Wronskian of the zeroth-order Bessel functions evaluated at $(k_i r)$, and is given in [8] from [9] as $2/(\pi(k_i r))$; hence

$$\left.\begin{aligned} &(k_i r)Z_1(k_i r) \; = \; 2/\left[\pi Y_0(k_i r)\right] \\ &\text{and similarly} \\ &(k_i R)Z_1(k_i R) \; = \; 2/\left[\pi Y_0(k_i R)\right] \end{aligned}\right\} \tag{31}$$

Let us now write

$$\hat{\alpha}_i(k_i r)Z_1(k_i r) \; = \; -\,\alpha_i\hat{\alpha}_0 \, .$$

Substituting this and eqns (31) into eqn (30) and dividing by $\hat{\alpha}_0^2$ gives

$$Y_N \ell n^2(R/r) \; = \; 2\pi j\omega\epsilon_A \sum_{i=1}^{N} \alpha_i^2 \left[\frac{1}{2\gamma_i}\right]\left[\left[\frac{Y_0(k_i r)}{Y_0(k_i R)}\right]^2 - 1\right]$$

$$+ \; 2\pi j\omega\epsilon_B \left[I'_{00} - 2\sum_{m=1}^{N} \alpha_m I'_{0m} + \sum_{m=1}^{N}\sum_{n=1}^{N} \alpha_m \alpha_n I'_{mn} \right] \tag{32}$$

where the integrals are now the limits as z' tends to zero of

$$I''_{00} \; = \; \int_0^\infty \exp\left[-z'(s^2 - k_B^2)^{\frac{1}{2}}\right] \frac{\left[J_0(sr) - J_0(sR)\right]^2}{s(s^2 - k_B^2)^{\frac{1}{2}}} \, ds$$

$$I''_{0m} = \int_0^{\infty} \exp\left[-z'(s^2-k_B^2)^{\frac{1}{2}}\right] \frac{s\left[J_0(sr)-J_0(sR)\right]\left[J_0(sr)-\left[Y_0(k_m r)/Y_0(k_m R)\right]J_0(sR)\right]}{(s^2 - k_B^2)^{\frac{1}{2}}(s^2 - k_m^2)}\, ds$$

$$I''_{mn} = \int_0^{\infty} \exp\left[-z'(s^2-k_B^2)^{\frac{1}{2}}\right]\left[\frac{s^3}{(s^2 - k_B^2)^{\frac{1}{2}}}\right]\left[\frac{J_0(sr)-\left[Y_0(k_m r)/Y_0(k_m R)\right]J_0(sR)}{(s^2 - k_m^2)}\right] \cdot$$

$$\cdot \left[\frac{J_0(sr) - \left[Y_0(k_n r)/Y_0(k_n R)\right]J_0(sR)}{(s^2 - k_n^2)}\right] ds$$

According to the asymptotic formulae for Bessel's J_0 (eg. [9]), the factors $J_0(sr)$ and $J_0(sR)$ tend to zero like $s^{-\frac{1}{2}}$ as s tends to infinity; hence the integrands of all thse integrals tend to zero like $(\exp(-z's)(s^{-3}))$ when approaching their upper limits. This form is (algebraically) convergent even when z' is set equal to zero, so we can now take the limit and obtain

$$I'_{00} = \int_0^{\infty} \frac{\left[J_0(sr) - J_0(sR)\right]^2}{s(s^2 - k_B^2)^{\frac{1}{2}}}\, ds \tag{33a}$$

$$I'_{0m} = \int_0^{\infty} \frac{s\left[J_0(sr) - J_0(sR)\right]\left[J_0(sr) - \left[Y_0(k_m r)/Y_0(k_m R)\right]J_0(sR)\right]}{(s^2 - k_B^2)^{\frac{1}{2}}(s^2 - k_m^2)}\, ds \tag{33b}$$

$$I'_{mn} = \int_0^{\infty} \frac{s^3\left[J_0(sr)-\left[Y_0(k_m r)/Y_0(k_m R)\right]J_0(sR)\right]\left[J_0(sr)-\left[Y_0(k_n r)/Y_0(k_n R)\right]J_0(sR)\right]}{(s^2 - k_B^2)^{\frac{1}{2}}(s^2 - k_m^2)(s^2 - k_n^2)}\, ds \tag{33c}$$

The integrands of these integrals have several false singularities and (in certain circumstances) one true one. We shall discuss these later; first we complete the treatment of eqn (32), which is now an easy matter. Taking the partial derivatives of Y_N with respect to each of the α's gives us, on equating them to zero and remembering that $I'_{mn} = I'_{nm}$,

$$\sum_{n=1}^{N} \alpha_n I'_{mn} + \alpha_m\left[\frac{\epsilon_A}{\epsilon_B}\right]\left[\frac{1}{2\gamma_m}\right]\left[\left[\frac{Y_0(k_m r)}{Y_0(k_m R)}\right]^2 - 1\right] = I'_{0m} \tag{34}$$

$$(m = 1, \ldots, N)$$

These N simultaneous linear equations in the N unknowns α_m determine the α_m of the N^{th} order approximation, and the corresponding admittance Y_N is then obtained from eqn

(32). One further simplification is possible; if we multiply each of eqns (34) by its corresponding α_m and sum over all m, we obtain

$$\sum_{m=1}^{N} \sum_{n=1}^{N} \alpha_m \alpha_n I'_{mn} + \left[\frac{\epsilon_A}{\epsilon_B} \right] \sum_{m=1}^{N} \alpha_m^2 \left[\frac{1}{2\gamma_m} \right] \left[\left[\frac{Y_0(k_m r)}{Y_0(k_m R)} \right]^2 - 1 \right]$$

$$= \sum_{m=1}^{N} \alpha_m I'_{0m}$$

and so eqn (32) reduces to

$$Y_N = \frac{2\pi j\omega\epsilon_B}{\ell n^2(R/r)} \left[I'_{00} - \sum_{m=1}^{N} \alpha_m I'_{0m} \right] \tag{35}$$

It will be noticed that none of the coefficients or right-hand sides in these equations depend explicitly on N; hence, calculating the complete set for a given N provides nearly all the information needed to determine the approximate admittances for every smaller N. This is very useful, as it allows several Y_N to be economically calculated and then supplied to an accelerated-convergence routine for the extrapolation to infinite N.

SINGULAR BEHAVIOUR

At first sight, the integrands of eqns (33) become infinite on the path of integration, at $s = 0$ in I'_{00}, $s = k_m$ in I'_{0m} and at $s = k_m$ or k_n in I'_{mn}. In fact, the numerators of the integrands have compensating zeros at all these points, so these singularities are illusory. From eqn (5), or equivalently from eqns (8), (7) and (6), $(J_0(sr)-(Y_0(k_m r)/Y_0(k_m R))J_0(sR))$ has a zero at $s = k_m$ (and since J_0 is an even function there is also a zero at $s = -k_m$, although this is not directly relevant); similarly there is compensation at $s = k_n$ in I'_{mn}, even when $k_n = k_m$. However, since the zeros in the numerators of the integrands are generally simple, the integrands remain finite at the false singular points rather than vanishing at them; this behaviour presents a numerical problem which is discussed further below.

There is no such problem at $s = 0$. Since $(J_0(sr) - J_0(sR))$ is of order (s^2) in that neighbourhood, the integrand of I'_{00} has a triple zero, rather than a pole, at $s = 0$; the same is also true of the other integrands, although again this is not directly relevant since they do not have apparent poles.

It remains to consider the behaviour of the integrands near $s = k_B$. Eqns (33) were established with k_B taken as strictly complex, while s is purely real on the path of integration; so in principle there is no problem here either. However, when the medium in region B is lossless, it is to be expected that the behaviour is given by the limit of the behaviour with a slightly lossy medium as the losses tend to zero, so we must show that this limit is well-behaved when the singular point at $s = k_B$ moves onto the path of integration.

We begin by confirming that the singular square root is itself well-defined. The original definition, for strictly complex k_B, was that $(s^2 - k_B^2)^{\frac{1}{2}}$ tends to s as s tends to positive real infinity. Over the real half-range of s, from 0 to ∞, $(s^2 - k_B^2)$ has a fixed

positive imaginary part and a real part which is large and positive for large s, falling to zero as s decreases to (real part of $(k_B^2))^{\frac{1}{2}}$ and then decreasing further to −(real part of (k_B^2)). Accordingly, the phase angle of $(s^2 - k_B^2)^{\frac{1}{2}}$ starts infinitesimally greater than zero for large positive s, and with decreasing s it increases monotonically to ($\pi/2$ + phase (k_B)), where $(-\pi/2) <$ phase $(k_B) < 0$ since k_B has a positive real part and a negative imaginary one. In the limiting case, when the negative imaginary part of k_B tends to zero, the phase of $(s^2 - k_B^2)^{\frac{1}{2}}$ is zero for $s > k_B$ and jumps to $\pi/2$ for $s < k_B$, corresponding to a pure positive real for large s and a pure positive imaginary for small s.

The easiest way to prove now that the integrals converge for real k_B is by means of the transformation previously used which removes the singularity for all k_B. Recalling the treatment of eqn (24) above, let us write $\ell = (s^2 - k_B^2)^{\frac{1}{2}}$, where the definition of the square root has just been examined; then, for instance, eqn (33a) gives

$$I'_{00} = \int_{jk_B}^{(S^2-k_B^2)^{\frac{1}{2}}} \frac{\left[J_0\left[r\sqrt{\ell^2+k_B^2}\,\right] - J_0\left[R\sqrt{\ell^2+k_B^2}\,\right]\right]^2}{(\ell^2+k_B^2)}\, d\ell + \int_S^{\infty} \frac{\left[J_0(sr) - J_0(sR)\right]^2}{s(s^2 - k_B^2)^{\frac{1}{2}}}\, ds \tag{36}$$

where: S is some convenient fixed large positive real (greater than (real part of $(k_B^2))^{\frac{1}{2}}$), the square roots in the Bessel function arguments are trouble-free because J_0 is an even function, and the path of the first integral passes through $\ell = 0$ when k_B is real. (As in treating eqn (24), we use the differential transformation $s\,ds = \ell d\ell$.) The second integral here has already been shown to converge, having satisfactory behaviour as s tends to infinity. The integrand of the first integral is well-behaved over the entire finite complex ℓ plane, the apparent poles at $\ell = \pm jk_B$ being cancelled by (double) zeros of the numerator, and the range of integration is finite; hence this integral is also finite. The other I-integrals can be treated in the same way with the same result; their apparent difficulties at $s = \pm k_m$ become apparent difficulties at $\ell = \pm(k_m^2 - k_B^2)^{\frac{1}{2}}$ and are still compensated as discussed earlier in this section.

NUMERICAL EVALUATION

We must now consider how the I-integrals are to be evaluated in practice.

It is tempting to try to reduce them using contour integration, but there does not seem to be any satisfactory way of doing this. There is an extensive literature on the Sommerfeld problem already mentioned (see [14]) which leads to integrals of somewhat similar form; but all the techniques employed with them succeed essentially because their integrands depend linearly on Bessel's J_0, and not on J_0^2 as in the present problem. We must therefore use an essentially numerical approach.

We consider first the problem of the infinite upper limit of integration in eqns (33). The parts of the integrands involving Bessel functions may all be written as

$$\left[J_0(sr) - y_m J_0(sR)\right]\left[J_0(sR) - y_n J_0(sR)\right]$$

where: m and n are the indices used in eqns (33), y_0 is 1, and $y_m = Y_0(k_m r)/Y_0(k_m R)$ for non-zero m. For large s, this expression may be reduced using the asymptotic expansion of J_0 from [11]; thus

$$J_0(x) = \left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \left\{\cos\left[x - \frac{1}{4}\pi\right] + \frac{1}{8x}\sin\left[x - \frac{1}{4}\pi\right] + 0\left[x^{-2}\right]\right\} \tag{37}$$

and so, neglecting terms of third and higher inverse orders,

$$\begin{aligned}
&\left[J_0(sr) - y_m J_0(sR)\right]\left[J_0(sr) - y_n J_0(sR)\right] \\
&= \frac{2}{\pi}\left\{\left[\frac{1}{sr}\right]^{\frac{1}{2}}\left[\cos\left[sr - \frac{1}{4}\pi\right] + \left[\frac{1}{8sr}\right]\sin\left[sr - \frac{1}{4}\pi\right]\right.\right. \\
&\qquad \left.\left. - y_m\left[\frac{1}{sR}\right]^{\frac{1}{2}}\left[\cos\left[sR - \frac{1}{4}\pi\right] + \left[\frac{1}{8sR}\right]\sin\left[sR - \frac{1}{4}\pi\right]\right]\right\} \cdot \\
&\qquad \cdot \left\{\left[\frac{1}{sr}\right]^{\frac{1}{2}}\left[\cos\left[sr - \frac{1}{4}\pi\right] + \left[\frac{1}{8sr}\right]\sin\left[sr - \frac{1}{4}\pi\right]\right]\right. \\
&\qquad \left. - y_n\left[\frac{1}{sR}\right]^{\frac{1}{2}}\left[\cos\left[sR - \frac{1}{4}\pi\right] + \left[\frac{1}{8sR}\right]\sin\left[sR - \frac{1}{4}\pi\right]\right]\right\} \\
&= \frac{2}{\pi}\left\{\left[\frac{1}{sr}\right]\left[\cos^2\left[sr - \frac{1}{4}\pi\right] + \left[\frac{1}{4sr}\right]\sin\left[sr - \frac{1}{4}\pi\right]\cos\left[sr - \frac{1}{4}\pi\right]\right]\right. \\
&\qquad + \left[\frac{y_m y_n}{sR}\right]\left[\cos^2\left[sR - \frac{1}{4}\pi\right] + \left[\frac{1}{4sR}\right]\sin\left[sR - \frac{1}{4}\pi\right]\cos\left[sR - \frac{1}{4}\pi\right]\right] \\
&\qquad - \left[\frac{y_m + y_n}{s(rR)^{\frac{1}{2}}}\right]\left[\cos\left[sr - \frac{1}{4}\pi\right]\cos\left[sR - \frac{1}{4}\pi\right]\right. \\
&\qquad\qquad + \left[\frac{1}{8sr}\right]\sin\left[sr - \frac{1}{4}\pi\right]\cos\left[sR - \frac{1}{4}\pi\right] \\
&\qquad\qquad \left.\left. + \left[\frac{1}{8sR}\right]\sin\left[sR - \frac{1}{4}\pi\right]\cos\left[sr - \frac{1}{4}\pi\right]\right]\right\} \\
&= \frac{1}{\pi}\left\{\left[\frac{1}{sr}\right]\left[1 + \cos\left[2sr - \frac{1}{2}\pi\right] + \left[\frac{1}{4sr}\right]\sin\left[2sr - \frac{1}{2}\pi\right]\right]\right.
\end{aligned}$$

$$+ \left[\frac{y_m y_n}{sR}\right]\left[1 + \cos\left[2sR - \frac{1}{2}\pi\right] + \left[\frac{1}{4sR}\right]\sin\left[2sR - \frac{1}{2}\pi\right]\right]$$

$$- \left[\frac{y_m+y_n}{s(rR)^{\frac{1}{2}}}\right]\left[\cos\left[s(r + R) - \frac{1}{2}\pi\right] + \cos\left[s(R - r)\right]\right.$$

$$+ \left[\frac{1}{8sr}\right]\left[\sin\left[s(r + R) - \frac{1}{2}\pi\right] - \sin\left[s(R - r)\right]\right]$$

$$\left.\left. + \left[\frac{1}{8sR}\right]\left[\sin\left[s(r + R) - \frac{1}{2}\pi\right] + \sin\left[s(R - r)\right]\right]\right]\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{1}{sr}\right]\left[1 + \sin(2sr) - \left[\frac{1}{4sr}\right]\cos(2sr)\right]\right.$$

$$+ \left[\frac{y_m y_n}{sR}\right]\left[1 + \sin(2sR) - \left[\frac{1}{4sR}\right]\cos(2sR)\right]$$

$$- \left[\frac{y_m+y_n}{s(rR)^{\frac{1}{2}}}\right]\left[\sin\left[s(R+r)\right]+\cos\left[s(R-r)\right]-\left[\frac{1}{8sr}\right]\left[\cos\left[s(R+r)\right]+\sin\left[s(R-r)\right]\right]\right.$$

$$\left.\left. - \left[\frac{1}{8sR}\right]\left[\cos\left[s(R + r)\right] - \sin\left[s(R - r)\right]\right]\right]\right\} \qquad (38)$$

The remaining factors in the integrands of eqns (33) may be written using the first-order binomial theorem as

$$\left[\frac{1}{s^2}\right]\left[1 + \left[\frac{1}{s^2}\right]\left[\frac{1}{2}k_B^2 + k_m^2 + k_n^2\right]\right] \qquad (39)$$

where $k_0 = 0$. At first sight this expression is inconsistent with eqn (37), in which terms of two inverse orders above the principal terms were neglected. However, the development of eqn (37) proceeds by powers of $(sr)^{-1}$ and $(sR)^{-1}$. These are of comparable magnitude, since R is roughly (2r) for most practical cases, and (from [8]) the first mode constant k_1 is approximately $\pi/(R - r)$; so $(sr)^{-1}$ (which is larger than $(sR)^{-1}$ and therefore determines the rate of decrease of successive terms) is approximately $k_1/(\pi L)$, where L is the lowest value of s to which the asymptotic form is applied. We are therefore neglecting in eqn (37) terms of relative size $(k_1/(\pi L))^2$ but retaining terms of relative size $((\frac{1}{2}k_B^2 + k_m^2 + k_n^2)/L^2)$. This is quite reasonable; at the N^{th} order of approximation to the admittance, k_m and k_n run up to k_N, which is roughly (Nk_1), so the ratio of the two relative sizes is roughly $(\pi N)^2$ – a substantial number even for

typical N of 6 or 8. Also, k_B^2 can be much larger than k_1^2 for high-permittivity dielectrics, and moreover the imaginary part of k_B^2 affects the real part of the admittance (which is usually smaller than the imaginary part and more difficult to calculate accurately). It is therefore quite reasonable to use expression (39) with eqn (38); and on combining them and discarding some higher-order product terms, we obtain as the asymptotic form of the integrands of eqns (33) the expression

$$\left[\frac{1}{\pi s^3}\right]\Bigg\{\left[\frac{1}{r}\right]\Bigg[1 + \sin(2sr) - \left[\frac{1}{4sr}\right]\cos(2sr)$$

$$+\Big[1 + \sin(2sr)\Big]\left[\frac{1}{s^2}\right]\left[\frac{1}{2}k_B^2 + k_m^2 + k_n^2\right]\Bigg]$$

$$+ \left[\frac{y_m y_n}{R}\right]\Bigg[1 + \sin(2sR) - \left[\frac{1}{4sR}\right]\cos(2sR)$$

$$+ \Big[1 + \sin(2sR)\Big]\left[\frac{1}{s^2}\right]\left[\frac{1}{2}k_B^2 + k_m^2 + k_n^2\right]\Bigg]$$

$$- \left[\frac{y_m + y_n}{(rR)^{\frac{1}{2}}}\right]\Bigg[\sin\Big[s(R + r)\Big] + \cos\Big[s(R - r)\Big]$$

$$-\left[\frac{1}{8sr}\right]\Big[\cos\big[s(R + r)\big] + \sin\big[s(R - r)\big]\Big]$$

$$-\left[\frac{1}{8sR}\right]\Big[\cos\big[s(R + r)\big] - \sin\big[s(R - r)\big]\Big]$$

$$+\Big[\sin\big[s(R + r)\big] + \cos\big[s(R - r)\big]\Big]\left[\frac{1}{s^2}\right]\left[\frac{1}{2}k_B^2 + k_m^2 + k_n^2\right]\Bigg]\Bigg\} \quad (40)$$

The integration of expression (40) between the lower limit L (already introduced) and the infinite upper limit is easily performed, using asymptotic integration for the trigonometric terms; eg.

$$\int_L^\infty \frac{\sin x}{x^i}\,dx = \left[-\frac{\cos x}{x^i}\right]_L^\infty - \int_L^\infty \frac{i\cos x}{x^{i+1}}\,dx$$

$$= \frac{\cos L}{L^i} - \left[\frac{i\sin x}{x^{i+1}}\right]_L^\infty - \int_L^\infty \frac{i(i+1)\sin x}{x^{i+2}}\,dx$$

$$= \frac{\cos L}{L^i} + \frac{i\sin L}{L^{i+1}} + 0\left[L^{-i-2}\right]$$

Hence $\int_L^{\infty} \left\{ \text{expression (40)} \right\} ds$

$$
\begin{aligned}
= \left[\frac{1}{\pi}\right] \Bigg\{ & \frac{1}{2rL^2} + \frac{\cos(2Lr)}{2r^2L^3} + \frac{3\sin(2Lr)}{4r^3L^4} + \frac{\sin(2Lr)}{8r^3L^4} \\
& + \left[\frac{1}{4rL^4} + \frac{\cos(2Lr)}{2r^2L^5}\right]\left[\frac{1}{2}k_B^2 + k_m^2 + k_n^2\right] \\
+ y_m y_n & \Bigg[\frac{1}{2RL^2} + \frac{\cos(2LR)}{2R^2L^3} + \frac{7\sin(2LR)}{8R^3L^4} \\
& + \left[\frac{1}{4RL^4} + \frac{\cos(2LR)}{2R^2L^5}\right]\left[\frac{1}{2}k_B^2 + k_m^2 + k_n^2\right]\Bigg] \\
- (y_m + y_m) & \Bigg[\frac{\cos(L(R+r))}{(rR)^{\frac{1}{2}}(R+r)L^3} + \frac{3\sin(L(R+r))}{(rR)^{\frac{1}{2}}(R+r)^2L^4} \\
& - \frac{\sin(L(R-r))}{(rR)^{\frac{1}{2}}(R-r)L^3} + \frac{3\cos(L(R-r))}{(rR)^{\frac{1}{2}}(R-r)^2L^4} \\
& + \frac{\sin(L(R+r))}{8(rR)^{\frac{1}{2}}r(R+r)L^4} - \frac{\cos(L(R-r))}{8(rR)^{\frac{1}{2}}r(R-r)L^4} \\
& + \frac{\sin(L(R+r))}{8(rR)^{\frac{1}{2}}R(R+r)L^4} + \frac{\cos(L(R-r))}{8(rR)^{\frac{1}{2}}R(R-r)L^4} \\
+ & \left[\frac{\cos(L(R+r))}{(rR)^{\frac{1}{2}}(R+r)L^5} - \frac{\sin(L(R-r))}{(rR)^{\frac{1}{2}}(R-r)L^5}\right]\left[\frac{1}{2}k_B^2 + k_m^2 + k_n^2\right]\Bigg]\Bigg\}
\end{aligned}
\tag{41}
$$

where some higher terms have been neglected. The orders of approximation in eqn (41) have been chosen to give satisfactory relative accuracy (a few parts per million) for N about 6 or 8, R comparable with (2r), L about k_{60} (note that L is not dimensionless) and the magnitude of ϵ_B (relative) less than about 1000 at the highest frequency at which the coaxial transmission-line mode propagates alone (the cut-off frequencies of the various kinds of higher mode are discussed in [8]). (The magnitude of the smallest of the complete integrals in eqns (33) is readily assessed to be $0(1/(k_N^2R))$, and this quantity is used as the reference.)

To complete the integrations in eqns (33), we must also integrate from 0 to L. There are three problems associated with integration over this range: the integrands oscillate rapidly (the shortest period in s is (π/R), as is shown by the asymptotic analysis above, and this is of the order of k_1 while L is about k_{60}); most of the integrands reduce to 0/0 at $s = k_m$ and/or $s = k_n$; and the integrands all become very large, or even infinite, when s^2 equals the real part of k_B^2. We will discuss each of these problems separately.

The reduction to 0/0 is eliminated as follows. Using l'Hopital's theorem, we can evaluate each integrand accurately at the false singular point (or points) in terms of Bessel's J_1. Each integrand can also be directly evaluated at fixed points close to the false singular point, with a modest loss of significant figures (say, 3, if $s = 1.001\ k_m$ or $0.999\ k_m$). Within the range spanned by the fixed points we can then use interpolation which keeps the loss of significant figures to the level already accepted (say, 3) provided the order of interpolation is chosen with regard to the accuracy expected in the Bessel functions. In the author's program four points are used and differences up to the third are retained, corresponding to 12 significant figures in the raw Bessel functions.

In order to treat the other two problems (the rapid oscillations of the integrands and the behaviour when s^2 is near $k_B{}^2$) we must first consider how the numerical integration should be done. The integrands in eqns (33) are rather complicated functions of the integration variable s, which makes it desirable to keep to a minimum the number of evaluations of each integrand; accordingly, Gaussian integration was chosen, using a four-point routine for eighth-order accuracy. An elementary integrator of this kind is normally embedded in a controlling subroutine which calls it recursively (or in a loop), with successively smaller sub-ranges of integration until the desired accuracy is achieved. In the present problem, we can eliminate this sub-division by making a virtue out of necessity; since the shortest period in s is (π/R), it is convenient and satisfactory to use preset sub-ranges of length $(\pi/(4R))$, each of which can be treated to satisfactory accuracy using a single four-point elementary Gaussian integration. This method has the happy consequence that all the integrands in eqns (33) - $((N+1)(N+2)/2)$ of them, for the N^{th} approximation - are evaluated at the same values of s, so the values of $J_0(sr)$ and $J_0(sR)$ which appear in them can be determined once and for all; this leads to a very substantial saving of computation time, since these Bessel functions dominate the time required to evaluate an integrand.

It is easy to identify the preset sub-range within which s^2 becomes equal to the real part of $k_B{}^2$. This sub-range, its immediate neighbour above, and its immediate neighbour below (if the frequency is high enough for it to have a neighbour below), are collectively treated separately from all the other sub-ranges, thereby isolating the neighbourhood of the singular or near-singular point; this amount of isolation is sufficient to keep the magnitudes of the integrands within sensible bounds over the rest of the range 0 to L. Integration over the three special sub-ranges is performed using the elementary four-point Gaussian integrator adaptively (that is, with recursively-nested calls); this is slow, but is accurate and straightforward to program. The three sub-ranges are treated together as one, but the first of the internal adaptive subdivisions is always taken at the point $s = (\text{real part of } (k_B{}^2))^{\frac{1}{2}}$, which should guarantee that no attempt will be made to evaluate an infinite integrand when k_B is purely real; for further security the program changes a purely real k_B to one with a very small (negative) imaginary part.

A Gaussian integrator, whether adaptive or elementary, is normally designed to integrate a real function over a real range. It is a trivial matter, however, to extend this to the integration of a complex function of a real variable over a real range, since all the calculations performed during the integration are linear in the calculated values of the function. The integrands in eqns (33) are only complex on account of the factor $(s^2 - k_B{}^2)^{\frac{1}{2}}$ - which is, of course, complex for some values of s even if k_B is purely real - and this complex square root can be evaluated and stored, like the values of $J_0(sr)$ and $J_0(sR)$, for each of the repeated values of s as described above. (A complex square root can be extracted using two real square roots.)

The rest of the numerical mathematics is comparatively trivial (see [8]). For some convenient moderately-sized N (6 or 8, say) the I-integrals are calculated and their values used to determine the α's in eqn (34), from which Y_N follows using eqn (35). Using only eqns (34) and (35), without calculating any more I-integrals, we can also determine Y_i for i = 1 (or even 0) to i = (N-1). The resulting set of Y's can then be processed

by an accelerated-convergence routine (such as the one described in [8]) to find the true coaxial terminating admittance Y.

CONCLUSION

A rigorous theoretical treatment of the open-ended coaxial line with infinite flange has been presented, together with all the essential computational details required to calculate its equivalent admittance at the plane of the open end. The treatment requires the numerical evaluation of one-dimensional integrals only; there are no multiple integrals involved.

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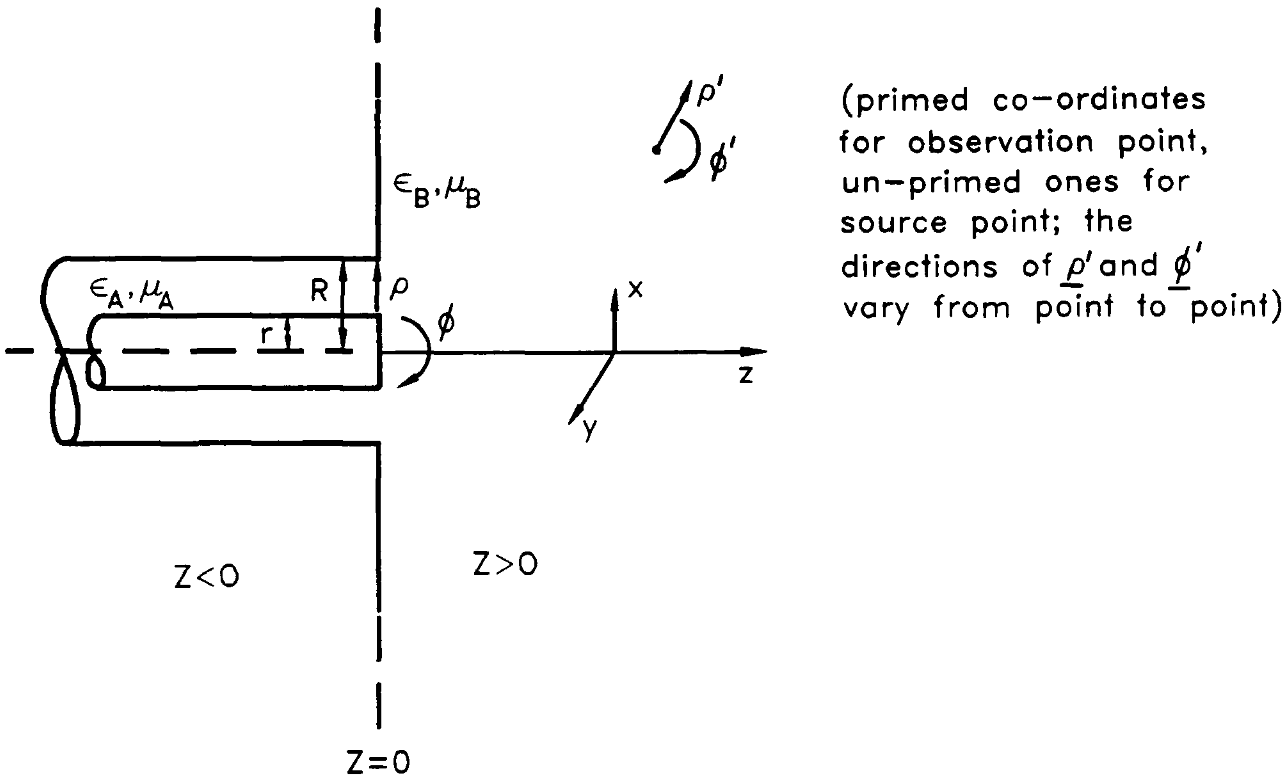
ϵ_B, μ_B
ϵ_A, μ_A
R
r
ρ
ϕ
x
z
y
ρ'
ϕ'
Z<0
Z>0
Z=0
(primed co-ordinates for observation point, un-primed ones for source point; the directions of $\underline{\rho}'$ and $\underline{\phi}'$ vary from point to point)

FIGURE.1

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Abstract

The open-ended coaxial line with a ground plane or effectively infinite flange has attracted much interest in recent years as a device for non-destructive measurement of complex permittivity, eg in bio-medical applications. Existing theoretical treatments of this configuration all involve approximations; this paper presents a rigorous treatment using a variational approach.

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